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A characterization of nonhomogeneous wavelet dual frames in Sobolev spaces

Jian-Ping Zhang* and Yun-Zhang Li

*Correspondence:
zhjp1982@emails.bjut.edu.cn
College of Applied Sciences, Beijing University of Technology, Beijing, 100124, China

Abstract

In recent years, nonhomogeneous wavelet frames have attracted some mathematicians' interest. This paper investigates such problems in a Sobolev space setting. A characterization of nonhomogeneous wavelet dual frames in Sobolev spaces pairs is obtained.

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Keywords: Bessel sequence; frame; nonhomogeneous wavelet dual frame; Sobolev spaces

1 Introduction

Wavelet frames in $L^2(\mathbb{R}^d)$ have been widely investigated by many authors [1–8]. In particular, homogeneous wavelet dual frames in $L^2(\mathbb{R}^d)$ were first characterized by Han [9], and then studied by Bownik [3]. For homogeneous wavelet dual frames, regularity and vanishing moments have been both required. However, for nonhomogeneous wavelet dual frames in Sobolev space pairs $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, they can be separated. It makes it easy to construct dual frames (see [10–14] for details). This paper is devoted to characterizing nonhomogeneous wavelet dual frames in Sobolev spaces pairs $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ via a pair of equations.

Before proceeding, we introduce some notions and notations. We denote by $\mathbb{Z}$ and $\mathbb{N}$ the set of integers and the set of positive integers, respectively. Let $d \in \mathbb{N}$. We denote by $\mathbb{T}^d = [0,1)^d$ the d-dimensional torus. For a Lebesgue measurable set E in $\mathbb{R}^d$, we denote by $|E|$ its Lebesgue measure and χ_E the characteristic function of E, respectively. And we write δ for the Dirac sequence, *i.e.*, $\delta_{0,0} = 1$ and $\delta_{0,k} = 0$ for $0 \neq k \in \mathbb{Z}^d$. The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is defined by

$$\hat{f}(\cdot) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \cdot \rangle}\, dx,$$

and extended to $L^2(\mathbb{R}^d)$ as usual, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$.

For $s \in \mathbb{R}$, we define Sobolev spaces $H^s(\mathbb{R}^d)$ as the space of all tempered distributions f such that

$$\|f\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left|\hat{f}(\xi)\right|^2 \left(1 + \|\xi\|^2\right)^s d\xi < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. The inner product in $H^s(\mathbb{R}^d)$ is given by

$$\langle f,g\rangle_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\bigl(1+\|\xi\|^2\bigr)^s\,d\xi, \quad f,g\in H^s\bigl(\mathbb{R}^d\bigr).$$

Moreover, for each $g\in H^{-s}(\mathbb{R}^d)$,

$$\langle f,g\rangle = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi, \quad f\in H^s\bigl(\mathbb{R}^d\bigr),$$

is a linear continuous functional in $H^s(\mathbb{R}^d)$. The $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ form pairs of dual spaces.

For functions $f,g:\mathbb{R}^d\mapsto\mathbb{C}$, define

$$[f,g]_t(\cdot) = \sum_{k\in\mathbb{Z}^d} f(\cdot+k)\overline{g(\cdot+k)}\bigl(1+\|\cdot+k\|^2\bigr)^t, \quad t\in\mathbb{R}.$$

For convenience, we write

$$f_{j,k}(\cdot) = 2^{\frac{jd}{2}} f\bigl(2^j\cdot-k\bigr) \quad\text{and}\quad f^s_{j,k}(\cdot) = 2^{j(\frac{d}{2}-s)} f\bigl(2^j\cdot-k\bigr)$$

for a distribution f, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^d$, and $s\in\mathbb{R}$.

Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Given $L\in\mathbb{N}$ and $s\in\mathbb{R}$, let $\phi,\psi_1,\psi_2,\ldots,\psi_L\in H^s(\mathbb{R}^d)$ and $\tilde{\phi},\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_L\in H^{-s}(\mathbb{R}^d)$, we denote by $X^s(\phi;\psi_1,\psi_2,\ldots,\psi_L)$ and $X^{-s}(\tilde{\phi};\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_L)$ the following two nonhomogeneous wavelet systems in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively:

$$X^s(\phi;\psi_1,\psi_2,\ldots,\psi_L) = \bigl\{\phi_{0,k}:k\in\mathbb{Z}^d\bigr\}\cup\bigl\{\psi^s_{l,j,k}:j\in\mathbb{N}_0,k\in\mathbb{Z}^d,l=1,2,\ldots,L\bigr\} \tag{1.1}$$

and

$$X^{-s}(\tilde{\phi};\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_L) = \bigl\{\tilde{\phi}_{0,k}:k\in\mathbb{Z}^d\bigr\}\cup\bigl\{\tilde{\psi}^{-s}_{l,j,k}:j\in\mathbb{N}_0,k\in\mathbb{Z}^d,l=1,2,\ldots,L\bigr\}. \tag{1.2}$$

We say that $X^s(\phi;\psi_1,\psi_2,\ldots,\psi_L)$ is a *nonhomogeneous wavelet frame* in $H^s(\mathbb{R}^d)$ if there exist two positive constants A, B such that

$$\begin{aligned} A\|f\|^2_{H^s(\mathbb{R}^d)} &\le \sum_{k\in\mathbb{Z}^d}\bigl|\langle f,\phi_{0,k}\rangle_{H^s(\mathbb{R}^d)}\bigr|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\bigl|\langle f,\psi^s_{l,j,k}\rangle_{H^s(\mathbb{R}^d)}\bigr|^2 \\ &\le B\|f\|^2_{H^s(\mathbb{R}^d)}, \quad \forall f\in H^s\bigl(\mathbb{R}^d\bigr), \end{aligned} \tag{1.3}$$

where A, B are called *frame bounds*; it is called a *nonhomogeneous wavelet Bessel sequence* in $H^s(\mathbb{R}^d)$ if the right-hand inequality in (1.3) holds, where B is called a *Bessel bound*. Furthermore, we say that $(X^s(\phi;\psi_1,\psi_2,\ldots,\psi_L), X^{-s}(\tilde{\phi};\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_L))$ is a pair of *nonhomogeneous wavelet dual frames* in $(H^s(\mathbb{R}^d),H^{-s}(\mathbb{R}^d))$ if $X^s(\phi;\psi_1,\psi_2,\ldots,\psi_L)$ and $X^{-s}(\tilde{\phi};\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_L)$ are Bessel sequences in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively, and

$$\langle f,g\rangle = \sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\phi}_{0,k}\rangle\langle\phi_{0,k},g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\bigl\langle f,\tilde{\psi}^{-s}_{l,j,k}\bigr\rangle\bigl\langle\psi^s_{l,j,k},g\bigr\rangle \tag{1.4}$$

holds for all $f\in H^s(\mathbb{R}^d)$ and $g\in H^{-s}(\mathbb{R}^d)$.

If $(X^s(\phi;\psi_1,\psi_2,\ldots,\psi_L), X^{-s}(\tilde{\phi};\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_L))$ is a pair of dual frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$, then it follows from (1.4) that

$$f = \sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\phi}_{0,k}\rangle\phi_{0,k} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\tilde{\psi}_{l,j,k}^{-s}\rangle\psi_{l,j,k}^{s}, \quad f\in H^s(\mathbb{R}^d),$$

and

$$g = \sum_{k\in\mathbb{Z}^d}\langle g,\phi_{0,k}\rangle\tilde{\phi}_{0,k} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle g,\psi_{l,j,k}^{s}\rangle\tilde{\psi}_{l,j,k}^{-s}, \quad g\in H^{-s}(\mathbb{R}^d),$$

with the series converging unconditionally in $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$, respectively.

The paper is organized as follows. Section 2 is devoted to some lemmas used latter. Section 3 is devoted to characterizing nonhomogeneous wavelet dual frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ via a pair of equations.

2 Some lemmas

In this section, we give some auxiliary lemmas which are necessary in proving Theorem 3.1 below.

Definition 2.1 Define a function $\kappa:\mathbb{Z}^d\to\mathbb{Z}$ by

$$\kappa(n) = \sup\{j\geq 0: 2^{-j}n\in\mathbb{Z}^d\}$$

for $0\neq n\in\mathbb{Z}^d$, and set $\kappa(0)=+\infty$.

Lemma 2.1 *Let $s\in\mathbb{R}$, $j\in\mathbb{Z}$, and $\psi\in H^{-s}(\mathbb{R}^d)$. Then, for $f\in H^s(\mathbb{R}^d)$ and $k\in\mathbb{Z}^d$, the kth Fourier coefficient of $[2^{\frac{jd}{2}}\hat{f}(2^j\cdot),\hat{\psi}(\cdot)]_0(\xi)$ is $\langle f,\psi_{j,k}\rangle$. In particular,*

$$[2^{\frac{jd}{2}}\hat{f}(2^j\cdot),\hat{\psi}(\cdot)]_0(\xi) = \sum_{k\in\mathbb{Z}^d}\langle f,\psi_{j,k}\rangle e^{2\pi i\langle k,\xi\rangle} \tag{2.1}$$

if $\{\psi_{j,k}:k\in\mathbb{Z}^d\}$ is a Bessel sequence in $H^{-s}(\mathbb{R}^d)$.

Proof Since $f\in H^s(\mathbb{R}^d)$ and $\psi\in H^{-s}(\mathbb{R}^d)$, we have $\hat{f}(2^j\cdot)\overline{\hat{\psi}(\cdot)}\in L^1(\mathbb{R}^d)$, and thus

$$\begin{aligned}\int_{\mathbb{T}^d}[2^{\frac{jd}{2}}\hat{f}(2^j\cdot),\hat{\psi}(\cdot)]_0(\xi)e^{-2\pi i\langle k,\xi\rangle}\,d\xi &= 2^{\frac{jd}{2}}\int_{\mathbb{T}^d}\sum_{l\in\mathbb{Z}^d}\hat{f}(2^j(\xi+l))\overline{\hat{\psi}(\xi+l)}e^{-2\pi i\langle k,\xi\rangle}\,d\xi\\ &= 2^{\frac{jd}{2}}\int_{\mathbb{R}^d}\hat{f}(2^j\xi)\overline{\hat{\psi}(\xi)}e^{-2\pi i\langle k,\xi\rangle}\,d\xi\\ &= 2^{-\frac{jd}{2}}\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{\psi}(2^{-j}\xi)}e^{-2\pi i\langle k,2^{-j}\xi\rangle}\,d\xi\\ &= \int_{\mathbb{R}^d}\hat{f}(\xi)\overline{[\psi_{j,k}(\cdot)]^{\wedge}(\xi)}\,d\xi,\end{aligned}$$

by the Plancherel theorem. So

$$\int_{\mathbb{T}^d}[2^{\frac{jd}{2}}\hat{f}(2^j\cdot),\hat{\psi}(\cdot)]_0(\xi)e^{-2\pi i\langle k,\xi\rangle}\,d\xi = \langle f,\psi_{j,k}\rangle. \tag{2.2}$$

If $\{\psi_{j,k} : k \in \mathbb{Z}^d\}$ is a Bessel sequence in $H^{-s}(\mathbb{R}^d)$, then $\{\langle f, \psi_{j,k}\rangle\}_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, and thus (2.1) follows by (2.2). □

By a careful observation of the proof of [13], Proposition 2.1, we have the following.

Lemma 2.2 *Let $s \in \mathbb{R}$, $\phi, \psi_1, \psi_2, \ldots, \psi_L \in H^s(\mathbb{R}^d)$. Then $X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ is a Bessel sequence in $H^s(\mathbb{R}^d)$ with Bessel bound B if and only if*

$$\sum_{k\in\mathbb{Z}^d} \left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \psi^s_{l,j,k}\rangle\right|^2 \le B\|g\|^2_{H^{-s}(\mathbb{R}^d)} \quad \textit{for } g \in H^{-s}\bigl(\mathbb{R}^d\bigr). \tag{2.3}$$

Lemma 2.3 *Let $s \in \mathbb{R}$, $\phi, \psi_1, \psi_2, \ldots, \psi_L \in H^s(\mathbb{R}^d)$. Suppose that $X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ is a Bessel sequence in $H^s(\mathbb{R}^d)$ with Bessel bound B, then*

$$\left|\hat{\phi}(\cdot)\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty} 2^{-2js}\left|\hat{\psi}_l\bigl(2^{-j}\cdot\bigr)\right|^2 \le B\bigl(1 + \|\cdot\|^2\bigr)^{-s} \tag{2.4}$$

holds a.e. on $\mathbb{R}^d$.

Proof Since $X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ is a Bessel sequence in $H^s(\mathbb{R}^d)$ with Bessel bound B, by Lemma 2.2, we have

$$\sum_{k\in\mathbb{Z}^d} \left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \psi^s_{l,j,k}\rangle\right|^2 \le B\|g\|^2_{H^{-s}(\mathbb{R}^d)} \quad \text{for } g \in H^{-s}\bigl(\mathbb{R}^d\bigr). \tag{2.5}$$

By Lemma 2.1 and an argument similar to that of [6], Theorem 1, we get

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \psi^s_{l,j,k}\rangle\right|^2 \\
&\quad= \int_{\mathbb{R}^d} \hat{\phi}(\xi)\overline{\hat{g}(\xi)} \sum_{k\in\mathbb{Z}^d} \hat{g}(\xi + k)\overline{\hat{\phi}(\xi + k)}\, d\xi \\
&\qquad+ \sum_{l=1}^{L}\sum_{j=0}^{\infty} 2^{-2js} \int_{\mathbb{R}^d} \hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{g}(\xi)} \sum_{k\in\mathbb{Z}^d} \hat{g}\bigl(\xi + 2^j k\bigr)\overline{\hat{\psi}_l\bigl(2^{-j}\xi + k\bigr)}\, d\xi.
\end{aligned}$$

It can be rewritten as

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \psi^s_{l,j,k}\rangle\right|^2 \\
&\quad= \int_{\mathbb{R}^d} \left|\hat{g}(\xi)\right|^2 \left(\left|\hat{\phi}(\xi)\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty} 2^{-2js}\left|\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\right|^2\right) d\xi \\
&\qquad+ \int_{\mathbb{R}^d} \overline{\hat{g}(\xi)} \sum_{0\ne k\in\mathbb{Z}^d} \hat{g}(\xi + k) \\
&\qquad\times \left(\hat{\phi}(\xi)\overline{\hat{\phi}(\xi + k)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)} 2^{-2js}\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{\psi}_l\bigl(2^{-j}(\xi + k)\bigr)}\right) d\xi
\end{aligned} \tag{2.6}$$

by the definition of κ.

Suppose (2.4) does not hold. Then there exists $E \subset \mathbb{R}^d$ with $|E| > 0$ such that

$$\left|\hat{\phi}(\cdot)\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty} 2^{-2js}\left|\hat{\psi}_l\left(2^{-j}\cdot\right)\right|^2 > B\left(1 + \|\cdot\|^2\right)^{-s} \quad \text{on } E,$$

and thus

$$\left|\hat{\phi}(\cdot)\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty} 2^{-2js}\left|\hat{\psi}_l\left(2^{-j}\cdot\right)\right|^2 > B\left(1 + \|\cdot\|^2\right)^{-s}$$

on some $E' = E \cap ([0,1)^d + k_0)$ with $|E'| > 0$ and $k_0 \in \mathbb{Z}^d$. Take g such that $\hat{g}(\cdot) = (1 + \|\cdot\|^2)^{s/2}\chi_{E'}$ in (2.6), then we obtain

$$\sum_{k\in\mathbb{Z}^d}\left|\langle g, \phi_{0,k}\rangle\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle g, \psi^s_{l,j,k}\rangle\right|^2 > B\left|E'\right| = B\|g\|^2_{H^{-s}(\mathbb{R}^d)},$$

contradicting (2.5). □

3 The characterization of nonhomogeneous wavelet dual frames in Sobolev spaces

This section is devoted to characterizing nonhomogeneous wavelet dual frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$. The following theorem provides us with a characterization via a pair of equations.

Theorem 3.1 *Let $s \in \mathbb{R}$, $\phi, \psi_1, \psi_2, \ldots, \psi_L \in H^s(\mathbb{R}^d)$ and $\tilde{\phi}, \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_L \in H^{-s}(\mathbb{R}^d)$. Define wavelet systems $X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ and $X^{-s}(\tilde{\phi}; \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_L)$ as in* (1.1) *and* (1.2), *respectively. Suppose that $X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ is a Bessel sequence in $H^s(\mathbb{R}^d)$, and $X^{-s}(\tilde{\phi}; \tilde{\psi}_1, \ldots, \tilde{\psi}_L)$ is a Bessel sequence in $H^{-s}(\mathbb{R}^d)$. Then $(X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L), X^{-s}(\tilde{\phi}; \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_L))$ is a pair of dual frames in $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ if and only if, for every $k \in \mathbb{Z}^d$,*

$$\hat{\phi}(\cdot)\overline{\hat{\tilde{\phi}}(\cdot + k)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)} \hat{\psi}_l\left(2^{-j}\cdot\right)\overline{\hat{\tilde{\psi}}_l\left(2^{-j}(\cdot + k)\right)} = \delta_{0,k} \quad \textit{a.e. on } \mathbb{R}^d. \tag{3.1}$$

Proof By the definition, $(X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L), X^{-s}(\tilde{\phi}; \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_L))$ is a pair of dual frames for $(H^s(\mathbb{R}^d), H^{-s}(\mathbb{R}^d))$ if and only if

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\langle f, \tilde{\phi}_{0,k}\rangle\langle\phi_{0,k}, g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f, \tilde{\psi}^{-s}_{l,j,k}\rangle\langle\psi^s_{l,j,k}, g\rangle \\ &\quad = \langle f, g\rangle, \quad f \in H^s\left(\mathbb{R}^d\right), g \in H^{-s}\left(\mathbb{R}^d\right).\end{aligned} \tag{3.2}$$

By the Plancherel theorem and Lemma 2.1, we deduce that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\langle f, \tilde{\phi}(\cdot - k)\rangle\langle\phi(\cdot - k), g\rangle + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f, \tilde{\psi}^{-s}_{l,j,k}\rangle\langle\psi^s_{l,j,k}, g\rangle \\ &\quad = \int_{\mathbb{T}^d}\left(\sum_{k\in\mathbb{Z}^d}\hat{f}(\xi + k)\overline{\hat{\tilde{\phi}}(\xi + k)}\right)\left(\sum_{k\in\mathbb{Z}^d}\hat{\phi}(\xi + k)\overline{\hat{g}(\xi + k)}\right)d\xi\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{L}\sum_{j=0}^{\infty}2^{jd}\int_{\mathbb{T}^d}\left(\sum_{k\in\mathbb{Z}^d}\hat{f}\bigl(2^j(\xi+k)\bigr)\overline{\hat{\tilde{\psi}}_l(\xi+k)}\right)\left(\sum_{k\in\mathbb{Z}^d}\hat{\psi}_l(\xi+k)\overline{\hat{g}\bigl(2^j(\xi+k)\bigr)}\right)d\xi\\
&=\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\hat{f}(\xi+k)\overline{\hat{\tilde{\phi}}(\xi+k)}\hat{\phi}(\xi)\overline{\hat{g}(\xi)}\,d\xi\\
&+\sum_{l=1}^{L}\sum_{j=0}^{\infty}2^{jd}\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\hat{f}\bigl(2^j(\xi+k)\bigr)\overline{\hat{\tilde{\psi}}_l(\xi+k)}\hat{\psi}_l(\xi)\overline{\hat{g}\bigl(2^j\xi\bigr)}\,d\xi\\
&=\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\left(\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi)}+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{\tilde{\psi}}_l\bigl(2^{-j}\xi\bigr)}\right)d\xi\\
&+\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\left(\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi+k)\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi+k)}\right.\\
&\left.+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}\bigl(\xi+2^jk\bigr)\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{\tilde{\psi}}_l\bigl(2^{-j}\xi+k\bigr)}\right)d\xi\\
&=\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\left(\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi)}+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{\tilde{\psi}}_l\bigl(2^{-j}\xi\bigr)}\right)d\xi\\
&+\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi+k)\left(\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi+k)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{\tilde{\psi}}_l\bigl(2^{-j}(\xi+k)\bigr)}\right)d\xi.
\end{aligned}
$$

And thus (3.2) can be rewritten as

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\left(\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi)}+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{\tilde{\psi}}_l\bigl(2^{-j}\xi\bigr)}\right)d\xi\\
&\quad+\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi+k)\left(\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi+k)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\bigl(2^{-j}\xi\bigr)\overline{\hat{\tilde{\psi}}_l\bigl(2^{-j}(\xi+k)\bigr)}\right)d\xi\\
&=\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.
\end{aligned}
\tag{3.3}
$$

Obviously, (3.1) implies (3.3). To finish the proof, next we prove the converse implication.

Suppose (3.3) holds. By Lemma 2.3 and the Cauchy-Schwarz inequality, the series

$$
\hat{\phi}(\cdot)\overline{\hat{\tilde{\phi}}(\cdot+k)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_l\bigl(2^{-j}\cdot\bigr)\overline{\hat{\tilde{\psi}}_l\bigl(2^{-j}(\cdot+k)\bigr)}
$$

with $k\in\mathbb{Z}^d$ converges absolutely a.e. on $\mathbb{R}^d$ and belongs to $L^\infty(\mathbb{R}^d)$, and almost all points in $\mathbb{R}^d$ are Lebesgue points. Let $\xi_0\in\mathbb{R}^d$ be such a point. For $0<\epsilon<\frac{1}{2}$, take f and g such that

$$
\hat{f}(\cdot)=\frac{(1+\|\cdot\|^2)^{-s/2}\chi_{B(\xi_0,\epsilon)}}{\sqrt{|B(\xi_0,\epsilon)|}}\quad\text{and}\quad\hat{g}(\cdot)=\frac{(1+\|\cdot\|^2)^{s/2}\chi_{B(\xi_0,\epsilon)}}{\sqrt{|B(\xi_0,\epsilon)|}}
$$

in (3.3), where $B(\xi_0,\epsilon) = \{\xi \in \mathbb{R}^d : |\xi - \xi_0| < \epsilon\}$. Then

$$\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\left(\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l(2^{-j}\xi)\overline{\hat{\tilde{\psi}}_l(2^{-j}\xi)}\right)d\xi = 1,$$

letting $\epsilon \to 0$ and applying the Lebesgue differentiation theorem, we obtain

$$\hat{\phi}(\xi_0)\overline{\hat{\tilde{\phi}}(\xi_0)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_l(2^{-j}\xi_0)\overline{\hat{\tilde{\psi}}_l(2^{-j}\xi_0)} = 1.$$

For $0 \neq k_0 \in \mathbb{Z}^d$, take f and g such that

$$\hat{f}(\cdot + k_0) = \frac{(1+\|\cdot\|^2)^{-s/2}\chi_{B(\xi_0,\epsilon)}}{\sqrt{|B(\xi_0,\epsilon)|}} \quad \text{and} \quad \hat{g}(\cdot) = \frac{(1+\|\cdot\|^2)^{s/2}\chi_{B(\xi_0,\epsilon)}}{\sqrt{|B(\xi_0,\epsilon)|}}$$

in (3.3), where $0 < \epsilon < \frac{1}{2}$. Then

$$\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\left(\hat{\phi}(\xi)\overline{\hat{\tilde{\phi}}(\xi+k_0)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k_0)}\hat{\psi}_l(2^{-j}\xi)\overline{\hat{\tilde{\psi}}_l(2^{-j}(\xi+k_0))}\right)d\xi = 0,$$

letting $\epsilon \to 0$ and applying the Lebesgue differentiation theorem, we obtain

$$\hat{\phi}(\xi_0)\overline{\hat{\tilde{\phi}}(\xi_0+k_0)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k_0)}\hat{\psi}_l(2^{-j}\xi_0)\overline{\hat{\tilde{\psi}}_l(2^{-j}(\xi_0+k_0))} = 0.$$

By the arbitrariness of ξ_0 and k_0, we obtain (3.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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